Hilbert Spaces

Inner Product Spaces (IPS)

Let $L(F)$ be a linear space, where F is field of scalars. $F = \mathbb{R}$ or $\mathbb{C}$.

An inner product on L is a mapping from $L \times L$ into F which assigns to each pair of vectors x, y in L, a scalar (x, y) in F s.t.

(1) $\overline{(x, y)} = (x, y)$; here $\overline{(x,y)}$ is complex conjugate of the numbers (x, y)

(2) $(\alpha x + \beta y, z) = \alpha(x, z) + \beta(y, z)$ → linearity

(3) $(x, x) \geq 0$, and $(x, x) = 0 \Leftrightarrow x = 0$
non-negativity

$\forall\, x, y, z \in L$ and $\alpha, \beta \in F$

(x, y) → inner product of x & y.

Hilbert Spaces Let H be a complex Banach space. Then H is called a Hilbert space if a complex numbers (x, y), called the inner product of x & y, is associated to each of the two vectors x & y in such a way that –

(1) $\overline{(x, y)} = (y, x)$

(2) $(\alpha x + \beta y, z) = \alpha(x, z) + \beta(y, z)$

(3) $(x, x) = \|x\|^2$ $\quad \forall\, x, y, z \in H$

& for all scalars α & β.

Examples

1. Consider $l_2^n \to$ Banach space
norm of $x \in l_2^n$ is defined as —

$$\|x\| = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2}$$

We shall show that if the inner product of two vectors $x = (x_1, x_2, .., x_n)$ & $y = (y_1, y_2, .., y_n)$ is defined by —

$$(x, y) = \sum_{i=1}^{n} x_i \bar{y}_i$$

then l_2^n is a Hilbert Space (HS)

Let $x, y, z \in l_2^n$

Then $x = (x_1, x_2 .., x_n)$, $y = (y_1, y_2 .., y_n)$,
$z = (z_1, z_2 . . , z_n)$

Let α, β be any scalars.

(1) $$\overline{(x, y)} = \overline{\left(\sum_{i=1}^{n} x_i \bar{y}_i \right)} = \overline{(x_1 \bar{y}_1 + \cdots + x_n \bar{y}_n)}$$
$$= \overline{(x_1, \bar{y}_1)} + \cdots + \overline{(x_n, \bar{y}_n)}$$
$$= \bar{x}_1 y_1 + \cdots + \bar{x}_n y_n$$
$$= y_1 \bar{x}_1 + \cdots + y_n \bar{x}_n$$
$$= \sum_{i=1}^{n} y_i \bar{x}_i$$
$$= (y, x)$$

(2) $$\alpha x + \beta y = \alpha(x_1, .., x_n) + \beta(y_1, .., y_n)$$
$$= (\alpha x_1 + \beta y_1, \ldots, \alpha x_n + \beta y_n)$$

$$\therefore (\alpha x + \beta y, z) = (\alpha x_1 + \beta y_1) \bar{z}_1 + \cdots + (\alpha x_n + \beta y_n) \bar{z}_n$$
$$= \alpha (x_1 \bar{z}_1 + \cdots + x_n \bar{z}_n) + \beta (y_1 \bar{z}_1 + \cdots + y_n \bar{z}_n)$$

$\therefore (\alpha x+\beta y, z) = \alpha(x,z) + \beta(y,z)$

(3) $(x,x) = \sum_{i=1}^{n} x_i \bar{x}_i = \sum_{i=1}^{n} |x_i|^2 = \|x\|^2$

$\therefore \ l_2^n$ is a H.S.

Properties of Hilbert Spaces: In a H.S., we have—

(i) $(\alpha x - \beta y, z) = \alpha(x,z) - \beta(y,z)$

(ii) $(x, \beta y + \gamma z) = \bar{\beta}(x,y) + \bar{\gamma}(x,z)$

(iii) $(x, \beta y - \gamma z) = \bar{\beta}(x,y) - \bar{\gamma}(x,z)$

(iv) $(x,0) = 0 \ \forall x \in H$ & $(0,x) = 0 \ \forall x \in H$

Thm (Schwarz Inequality)

If x & y are any two vectors in a H.S. 'H' then $|(x,y)| \leq \|x\| \, \|y\|$

Pf: If $y=0$ then $\|y\|=0$ and $|(x,y)|=0$

$\therefore$ thm is true in this case.

Let $y \neq 0$. For any scalar λ, we have

$(x+\lambda y, x+\lambda y) \geq 0$, $\because (z,z) \geq 0 \ \forall z \in H$

$\Rightarrow (x, x+\lambda y) + \lambda(y, x+\lambda y) \geq 0$; linearity

$\Rightarrow (x,x) + (x, \lambda y) + \lambda(y,x) + \lambda(y, \lambda y) \geq 0$

$\Rightarrow \|x\|^2 + \bar{\lambda}(x,y) + \lambda(y,x) + \lambda\bar{\lambda}(y,y) \geq 0$

$\Rightarrow \|x\|^2 + \bar{\lambda}(x,y) + \lambda(y,x) + |\lambda|^2 \|y\|^2 \geq 0$ —(1)

$\because y \neq 0$, $\therefore \|y\| \neq 0$

Put $\lambda = -\dfrac{(x,y)}{\|y\|^2}$ in (1), we get—

$$\Rightarrow \|x\|^2 - \frac{\overline{(x,y)}}{\|y\|^2}(x,y) - \frac{(x,y)}{\|y\|^2}(y,x) + \frac{|(x,y)|^2}{(\|y\|^2)^2}\|y\|^2 \geq 0$$

$$\Rightarrow \|x\|^2 - \frac{|(x,y)|^2}{\|y\|^2} - \frac{\overline{(x,y)}\,(x,y)}{\|y\|^2} + \frac{|(x,y)|^2}{\|y\|^2} \geq 0$$

$$\Rightarrow \|x\|^2 - \frac{|(x,y)|^2}{\|y\|^2} - \frac{|(x,y)|^2}{\|y\|^2} + \frac{|(x,y)|^2}{\|y\|^2} \geq 0$$

$$\Rightarrow \|x\|^2 - \frac{|(x,y)|^2}{\|y\|^2} \geq 0$$

$$\Rightarrow \|x\|^2\|y\|^2 - |(x,y)|^2 \geq 0$$

$$\Rightarrow |(x,y)|^2 \leq \|x\|^2\|y\|^2$$

$$\Rightarrow |(x,y)| \leq \|x\|.\|y\|$$

(Proved)

Th^m. In a H.S., the inner product is jointly continuous

i.e. $x_n \to x,\ y_n \to y \Rightarrow (x_n, y_n) \to (x,y)$

Pf: we have,

$$|(x_n,y_n) - (x,y)| = |(x_n,y_n) - (x_n,y) + (x_n,y) - (x,y)|$$

$$= |(x_n, y_n - y) + (x_n - x, y)|$$

, by linearity

$$\leq |(x_n, y_n - y)| + |(x_n - x, y)|$$

$; \because |\alpha+\beta| \leq |\alpha| + |\beta|$

$$\leq \|x_n\|\|y_n - y\| + \|x_n - x\|\|y\| \quad \to (1)$$

Since $x_n \to x$, $y_n \to y$ for $n \to \infty$

$\therefore \|x_n - x\| \to 0$ & $\|y_n - y\| \to 0$ as $n \to \infty$

$\therefore$ From (1) $\to$

$$|(x_n, y_n) - (x,y)| \to 0 \text{ for } n \to \infty$$

Hence $(x_n, y) \to (x,y)$

(Proved)